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TENTAMEN RELATIVISTIC QUANTUM MECHANICS

Friday 12-11-2010, 09.00-12.00

On the first sheet write your name, address and student number. Write your name on all other sheets.

This examination consists of four problems, with in total 18 parts. The 18 parts carry equal weight in determining the final result of this examination.

$\hbar = c = 1$. The standard representation of the 4×4 Dirac gamma-matrices is given by:

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}.$$

PROBLEM 1

A solution of the Dirac equation for a free spinor field is of the form

$$\psi(x) = u(p,s)e^{-ipx}\Big|_{p^0=\omega_p}.$$

In this problem the momentum p is always on-shell, $p^0 = \omega_p$.

1.a Show that $\psi(x)$ is also a solution of the Klein-Gordon equation.

1.b The Dirac equation implies an equation for $u(p,s)$. Determine this equation.

1.c Show that the matrix γ^0 has two eigenvalues $+1$ and two eigenvalues -1 (do not use an explicit representation for the γ-matrices!).

1.d The solution for $u(p,s)$ is

$$u(p,s) = \frac{\gamma^\mu p_\mu + m}{\sqrt{2m(m+\omega_p)}}u(0,s)\,, \tag{1.1}$$

with $s = 1, 2$, and

$$\gamma^0 u(0,s) = u(0,s)\,. \tag{1.2}$$

Show that eq. (1.2) follows from eq. (1.1) in the restframe.

1.e Show that

$$\bar{u}(p,s)u(p,t) = \delta_{st}\,.$$

PROBLEM 2

Consider the following Lagrangian density for a quantum field theory involving two scalar fields ϕ_1 and ϕ_2:

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + a\partial_\mu\phi_1\partial^\mu\phi_2 + \tfrac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - 2m^2\phi_1\phi_2.$$

where a is a constant.

2.a Determine the equations of motion for ϕ_1 and ϕ_2.

2.b What are the canonical momenta π_1 and π_2 associated to ϕ_1 and ϕ_2?

2.c Express the Hamiltonian in terms of the time derivatives and spatial derivatives of the fields.

2.d Give the result of 2.c for the special case $a = 0$.

2.e Show that for $a = 1$ the equations of motion imply that $\phi_1 = \phi_2$. What is in this case the mass of the field $\phi \equiv \phi_1 + \phi_2$?

PROBLEM 3

The Lagrangian density for the Dirac field is

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) .$$

3.a Define the canonical momentum corresponding to the field ψ, and show that it equals

$$\pi(t, \vec{x}) = i\psi^\dagger(t, \vec{x}) .$$

3.b The Hamiltonian is defined as

$$H = \int d^3x \, (\pi(t, \vec{x})\partial_0\psi(t, \vec{x}) - \mathcal{L}) .$$

Show that this equals

$$H = -\int d^3x \, \bar{\psi}(i\gamma^k \partial_k - m)\psi .$$

if $\psi(x)$ satisfies the Dirac equation.

3.c The invariance of $\mathcal{L}$ under transformations

$$\psi \to \psi' = e^{i\theta}\psi$$

gives rise to a current $j^\mu \equiv \bar{\psi}\gamma^\mu\psi$. Show that, if the Dirac equation for ψ (and $\bar{\psi}$) holds, j^μ satisfies

$$\partial_\mu j^\mu = 0 .$$

3.d Show that

$$Q = \int d^3x \, j^0$$

is constant in time if ψ and its spatial derivatives go sufficiently fast to zero at large $|\vec{x}|$.

PROBLEM 4

Consider the scattering of a photon and a proton. The photon has energy E_1 and momentum $\vec{p}_1$, the proton of mass M is at rest. The spatial momentum of the photon is in the direction x^1. We will discuss only properties of the initial state.

4.a What are the components of the four-momenta of the photon and the proton?

4.b We perform a Lorentz transformation to the center-of mass frame, so that in the new system the total spatial momentum vanishes. In the new system the energy and momentum of the photon are E_3, p_3, of the proton E_4, p_4. Express E_3 and E_4 in terms of p_3 and M.

4.c The Lorentz transformation required to do this is of the form

$$\Lambda^0{}_0 = \Lambda^1{}_1 = \gamma\,,\ \Lambda^0{}_1 = \Lambda^1{}_0 = -\frac{\gamma v}{c}\,,\ \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\,.$$

Perform this Lorentz transformation on the initial four-momenta to express $E_3,\ p_3,\ E_4,\ p_4$ in terms of $v/c,\ E_1$ and M.

4.d Express v/c in terms of E_1 and M with the relations obtained in parts (4.b) and (4.c).

3